The stability of a shear layer in an unbounded heterogeneous inviscid fluid

By P. G. DRAZIN
Trinity College, Cambridge

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SUMMARY

This paper considers the stabilizing effect of density stratification on the horizontal shear layer between two parallel streams of uniform velocities. A simple continuous velocity distribution, $U = V\tanh(y/d)$, is used to represent the laminar shear layer. The density of the fluid is assumed to vary as $\exp(-\beta y)$, y being the vertical coordinate, with a small total change in density across the shear layer. The fluid is unbounded, and is assumed to be inviscid, incompressible and under the action of gravity.

By the methods of hydrodynamic stability theory, it is shown that a disturbance of small amplitude and wave-number α is neutrally stable if the Richardson number, defined as $J = g\beta d^2/V^2$, has the value $\alpha^2 d^2(1-\alpha^2 d^2)$, and the form of the neutral disturbance is obtained. It follows that the critical Richardson number is $\frac{1}{4}$, so that the flow is stable if $J > \frac{1}{4}$. The relation between these results and Goldstein's derivation of the same critical Richardson number for a flow with discontinuous velocity and density gradients is discussed.

1. INTRODUCTION

The natural occurrence of one parallel stream over another denser stream is widespread. Familiar examples are a river flowing into the sea and a warm layer of wind blowing over a cool one. The knowledge that the tendency of gravity to keep the denser fluid below the lighter might strongly inhibit turbulence in such flows aroused interest in their stability many years ago. This effect of gravity was first investigated by Stokes, who treated the stability of one fluid resting above another of greater density. Many authors extended Stokes's work to various flows of several superposed horizontal layers of fluids with piece-wise constant density and horizontal velocity (such that the density and velocity are constant in each layer but may vary from layer to layer). The stability of a heterogeneous fluid (i.e. a fluid with a continuous vertical variation of density) at rest also was treated by Lord Rayleigh and others.

The general problem of stability of an inviscid fluid with both density and velocity continuously varying with height was approached in 1931 by

Taylor and Goldstein independently. They each doubted the ability of a flow with several layers of homogeneous fluids to represent the phenomena occurring in the sea and atmosphere. The mathematical question was whether the stability of a heterogeneous fluid could be approximated by use of a large number of layers of homogeneous fluids. To study this question they derived the equation of hydrodynamic stability of small amplitude waves in a parallel primary flow of an inviscid heterogeneous fluid under gravity. They went on to consider the stability of some special flows with layers, in each of which either the shear (i.e. the velocity gradient) or the velocity is constant and the density either is constant or varies exponentially. They each concluded that a multi-layer system of homogeneous fluids could not be used to approximate the stability of a heterogeneous fluid.

In § 2 of this paper the stability equation of Taylor (1931) and Goldstein (1931) is derived in dimensionless form. In § 3 their results are summarized and discussed in the light of subsequent work. The natural phenomenon of a shear layer in a heterogeneous fluid is more accurately described in § 4 by use of continuously varying functions for velocity and density. This avoids the possible effect of discontinuities in either velocity or velocity gradient and either density or density gradient on the stability characteristics; also the vorticity (i.e. the velocity gradient) is not assumed constant. It is then shown simply that a small-amplitude wave disturbance of (dimensionless) wave-number α is neutrally stable if the Richardson number $J = \alpha^2(1-\alpha^2)$. It follows that the flow is stable for all wave disturbances if $J > \frac{1}{4}$, the same critical Richardson number found by Goldstein (1931) for a shear layer with two discontinuities in the primary vorticity.

2. The stability equation

Following Taylor (1931) and Goldstein (1931), we consider the inviscid stability equation for a parallel flow under gravity with a steady horizontal velocity $U(y)$ varying with height y. The fluid is heterogeneous with density $\bar{\rho}(y)$, although each material particle is incompressible.

First consider the dimensionless parameters of this primary flow. Suppose that the velocity field has characteristic scales d of length and V of velocity; and that the density distribution has length scale $1/\beta$. Then the flow may be typified by two dimensionless parameters. We define the Richardson number as

$$J = g\beta d^2/V^2, \tag{1}$$

which measures the ratio of buoyancy forces to inertia forces. The larger J the more stable the flow, because the steeper the density gradient the more energy of a disturbance is used to lift heavier fluid among lighter and push down lighter among heavier. We define the ratio of the length scale of velocity to that of density as

$$L = \beta d. \tag{2}$$

It can be seen from the equation of motion that L represents the effect of variation of inertia arising from heterogeneity. Note that $L = JF$, where the Froude number $F = V^2/gd$.

The vector form of Euler's equations of motion for a gravitating inviscid fluid is

$$\frac{D\mathbf{u}}{Dt} = -\frac{1}{\rho}\nabla p - g\nabla y. \tag{3}$$

The condition of incompressibility and the equation of continuity lead to

$$D\rho/Dt = 0, \tag{4}$$

and

$$\nabla . \mathbf{u} = 0. \tag{5}$$

Now consider the velocity field

$$u = U(y) + u', \qquad v = v', \quad w = 0,$$

the pressure

$$p = -g\int \bar{\rho}\, dy + p', \tag{6}$$

and the density

$$\rho = \bar{\rho}(y) + \rho',$$

representing the superposition of a primary flow and a small amplitude wave-disturbance (denoted by primes).

The disturbance is assumed to be two-dimensional (so that $w = 0$) on the basis of the work of Yih (1955). Yih showed essentially that, for all parallel flows of a viscous heterogeneous incompressible fluid, the stability characteristics of a three-dimensional wave-disturbance are the same as those of a two-dimensional one at higher Richardson number J and lower Reynolds number. Thus only two-dimensional disturbances need be considered in the search for a sufficient criterion of stability, with minimum J and maximum Reynolds number.

Any sufficiently small disturbance may be resolved into Fourier components which are dynamically independent, and so primed quantities will be taken as a typical component, being the real part of a complex quantity proportional to $\exp\{i\alpha(x-ct)\}$, where α is a positive wave-number and $c = c_r + ic_i$ is a complex velocity. This wave has a phase velocity c_r in the x-direction and logarithmic rate of increase αc_i in amplitude; thus $c_i = 0$ for neutral stability. Taylor (1931) and Goldstein (1931) eliminated u', p' and ρ' from the equations of motion to get the equation for v'. Here equivalently we use the stream function of the disturbance

$$\psi' = \phi(y)\exp\{i\alpha(x-ct)\}, \tag{7}$$

to integrate the equation of continuity with

$$u' = \partial\psi'/\partial y, \qquad v' = -\partial\psi'/\partial x = -i\alpha\psi'. \tag{8}$$

Then elimination of p' and ρ' from the equations of motion leads to the equation

$$(U-c)(\phi''-\alpha^2\phi) - U''\phi - \frac{g\bar{\rho}'/\bar{\rho}}{U-c}\phi + \frac{\bar{\rho}'}{\bar{\rho}}\{(U-c)\phi' - U'\phi\} = 0,$$

where primes now and henceforth denote differentiations with respect to y. This becomes Rayleigh's equation if the fluid is homogeneous, i.e. if $\bar{\rho}' = 0$. When the appropriate dimensional scales are divided out of each quantity, this equation takes the dimensionless form

$$(U-c)(\phi''-\alpha^2\phi)-U''\phi-\frac{J\bar{\rho}'/\beta\bar{\rho}}{U-c}\phi+L(\bar{\rho}'/\beta\bar{\rho})\{(U-c)\phi'-U'\phi\}=0. \quad (9)$$

Taylor and Goldstein neglected the terms in L, which come from the effect of heterogeneity on the inertial terms in the equations of motion. This simplifies the mathematics, and L is in fact small in cases of practical interest where the density variation across the whole shear layer is small. When $J \neq 0$, the type of the singularities of equation (9) is not altered by putting $L = 0$.

In the cases examined by Taylor and Goldstein the density is piece-wise or varies exponentially, i.e. $\bar{\rho}'/\bar{\rho} = -\beta$, where β is zero or constant in each layer. This restriction does not appear to alter any essential feature of the stability of the real flow in a shear layer. Equation (9) is thus transformed to

$$(U-c)(\phi''-\alpha^2\phi)-U''\phi+J\phi/(U-c)=0. \quad (10)$$

3. Discussion of the work of Taylor and Goldstein

Taylor (1931) and Goldstein (1931) solved equation (10) for two cases. For constant U there are two exponential solutions

$$\phi = \exp[\pm\{\alpha^2-J(U-c)^{-2}\}^{1/2}].$$

For linear U the solutions were found in simple terms of Bessel functions of order $\frac{1}{2}(1-4J)^{1/2}$ and complex argument. Taylor found

$$\phi \to (U-c)^{\frac{1}{2}\{1\pm\sqrt{(1-4J)}\}}$$

as $U \to c$, if U is linear near the critical plane where $U = c$. Therefore ϕ decreases to zero as $U \to c$ if $0 < J < \frac{1}{4}$; if $J > \frac{1}{4}$, then the singularity differs in that ϕ oscillates infinitely rapidly as well as decreases in amplitude. With either singularity the horizontal velocity of the disturbance (proportional to ϕ') is infinite.

Taylor's consideration of several problems of three and four superposed streams, each of homogeneous fluid, indicated that there might be stability for $J > \frac{1}{4}$ in the limiting case of a continuous density distribution. This indication was supported by observation of the behaviour of streamlines depending on the singularity of ϕ near the critical plane, yet was apparently contradicted by his results for a continuous density distribution. In spite of the difficulty of handling the Bessel functions, he solved the problem with linear velocity and exponential density above a rigid horizontal plane. Only neutral waves can exist if $J \equiv g\beta/(\text{velocity gradient})^2 > \frac{1}{4}$, and no waves at all can exist if $0 < J < \frac{1}{4}$.

The condition $J = \frac{1}{4}$ does not represent critical stability in the usual sense that all waves are stable for each $J > \frac{1}{4}$ and that there is at least one unstable wave for each $J < \frac{1}{4}$. Rather, his solution implies that the assumption of Fourier resolution of an arbitrary disturbance into separate wave components is invalid for $0 < J < \frac{1}{4}$ in the inviscid flow considered. In fact, the possibility of a *zero* or *negative* critical Richardson number in Taylor's semi-bounded flow can be shown by comparison with plane Couette flow between two rigid horizontal planes. Wasow (1953) proved that this flow of a homogeneous viscous fluid was stable to small disturbances at sufficiently large Reynolds numbers. This means that the critical Richardson number for the inviscid stability of bounded plane Couette flow is non-positive, because density increasing with height is needed to create instability. Thus there may be some stable disturbances in addition to those waves found by Taylor for $J > 0$.

Goldstein (1931) was principally concerned with a three-layer flow having $U = u_1$ and $\bar{\rho} = \rho_1 \exp(-2\beta h)$ for $y \geqslant h$; $U = u_1 y/h$ and $\bar{\rho} = \rho_1 \exp\{-\beta(y+h)\}$ for $h \geqslant y \geqslant -h$; and $U = -u_1$ and $\bar{\rho} = \rho_1$ for $-h \geqslant y$. He found that disturbances can be neutrally stable only if $J = g\beta h^2/u_1^2 \leqslant \frac{1}{4}$, and therefore the flow is stable or unstable according as $J > \frac{1}{4}$ or $J < \frac{1}{4}$ respectively.

Comparison of their results for piece-wise constant density with those for continuous density led Taylor and Goldstein each to conclude that not all the essential features of a continuous flow could be approximated in the limit by a multi-layer system. This conclusion was based on their particular results for three and four layers and on the doubt whether a multi-layer flow could approximate the singularity of the continuous flow in the critical plane where $U = c$.

In 1951 Scorer derived equation (9) from the equations for an n-layer system by letting $n \to \infty$. Also, Benton (1953) found the equation for the eigenvalues of the complex wave-velocity of an n-layer system and then let $n \to \infty$. For certain velocity and density distributions he found that the progressive wave-speeds were the same in the limit as those of the corresponding continuous flow. Recent authors have generally favoured the conclusion that the eigenvalues and eigenfunctions of a general continuous flow are identical with the limits of the corresponding values and functions in an n-layer flow, but no proof has been given yet. In any event the limit cannot be uniform in n and y near the critical plane in view of Taylor's observation that an n-layer system cannot have the infinite horizontal velocity associated with the singular behaviour of ϕ.

4. The stability of the flow with smoothly-varying velocity and density

Discontinuities of velocity and density gradients or of uniform shear, as in the flows considered by Taylor (1931) and Goldstein (1931), are discrepancies from the real flow which may influence the calculation of stability. It might be thought that any continuously curved velocity profile would be more difficult to handle than a profile consisting of straight line segments, but this is not always so. With a profile shaped like $\tanh y$ the stability equation (10) can be transformed to a second-order differential equation with four regular singularities. The antisymmetry of the profile then enables the stability criterion to be found for each wave-number without using any detailed properties of the general solution of the equation.

Following Curle's (1956) work on the corresponding flow of a homogeneous fluid, let

$$U = \tanh y \qquad (-\infty < y < \infty) \tag{11}$$

in dimensionless form. (This profile is shown in figure 1 (*a*).) Thus the velocity scale V is half the velocity difference across the shear layer. The length scale d is V divided by the velocity gradient at $y = 0$. Comparative studies of stability suggest that the exact shape of the velocity profile is not of much significance in stability criteria. However, the profile (11) can be well matched to that of a free boundary layer between parallel streams as calculated by Lock (1951), who supposed the streams meet in a line perpendicular to their flow and used Blasius's equation for the flow downstream of the line.

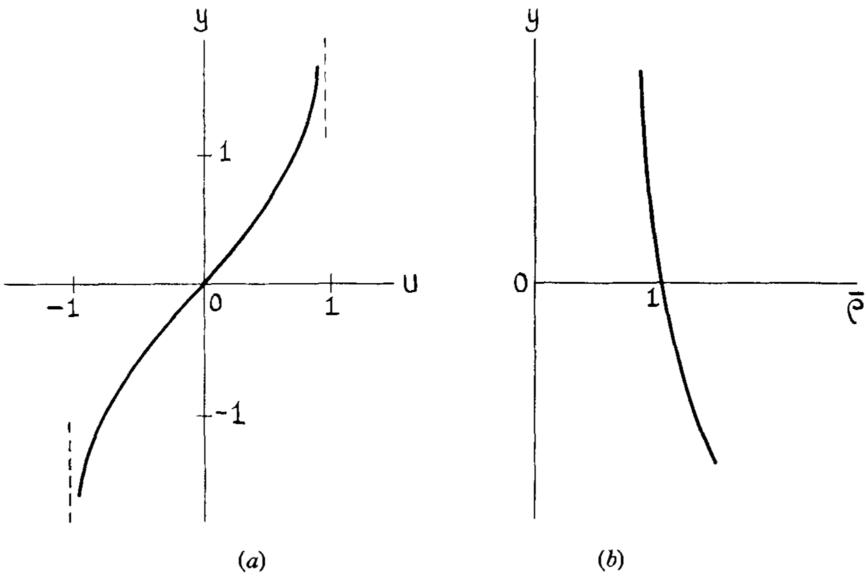

Figure 1. (*a*) The velocity profile $U = \tanh y$. (*b*) A typical profile of the density $\bar{\rho} = \exp(-Ly)$.

Let $$\bar{\rho} = \exp(-Ly) \qquad (-\infty < y < \infty) \tag{12}$$

in dimensionless form. The infinity of $\bar{\rho}$ at $y = -\infty$ and the zero at $y = +\infty$ lead to a wave exponentially damped at $y = \pm\infty$ where the primary flow is uniform, provided $\alpha > \frac{1}{2}L$ (as can be shown by examination of equation (9) in asymptotic form). There is no singular behaviour caused by the neglect of L, so equation (10) will be used here. A flow with negligible L is essentially equivalent to a real flow whose stability is determined by the wave-development in the interfacial shear layer, not in the surrounding uniform streams where all disturbances rapidly die away.

The inviscid boundary conditions that the vertical velocity vanishes at infinity become
$$\alpha\phi \to 0 \quad \text{as } y \to \pm\,\infty. \tag{13}$$
These conditions and equation (10) incidentally ensure that the horizontal velocity also vanishes at infinity.

Before solving the eigenvalue problem, we note its time and space symmetry. The equation and the boundary conditions are real because the inviscid flow is time-reversible. Therefore, if c is an eigenvalue corresponding to the eigenfunction ϕ for given values of α and J, $c^* = c_r - ic_i$ is an eigenvalue corresponding to the function ϕ^* for the same α and J. A nominal reverse of the y-direction changes the sign of g and β, but not of their product in J. Also it changes the sign of U, an odd function of y, but leaves the boundary conditions unchanged. There thus is no difference in the positive and negative x-directions in the specification of the problem. Therefore, if c is an eigenvalue corresponding to $\phi(y)$ for given α and J, $-c$ is an eigenvalue corresponding to $\phi(-y)$ for the same α and J. This gives, for each wave travelling in one direction, another similar wave in the opposite direction. On the grounds of uniqueness it is natural to suppose that, for a neutral disturbance with $c_i = 0$ at any rate, these waves coincide and the wave speed $c_r = 0$. This certainly is true for a neutral disturbance of the flow with uniform density, i.e. with $J = 0$, because Tollmien (1935) proved that $c = U$ where $U'' = 0$ for general monotonic U. The particular velocity profile $U = \tanh y$ is monotonic and is zero at the only point where $U'' = 0$, namely at $y = 0$. Therefore, in seeking a neutral wave with $c_i = 0$ for a criterion of stability of the flow of a heterogeneous fluid, it will be assumed that
$$c = 0. \tag{14}$$
This will be shown to lead to an eigensolution. The uniqueness of this solution will be supported in the first paragraph of § 5.

Thus equation (10) becomes
$$U(\phi'' - \alpha^2\phi) - U''\phi + J\phi/U = 0, \tag{15}$$
i.e.
$$\phi'' + (2\,\mathrm{sech}^2 y - \alpha^2 + J\coth^2 y)\phi = 0. \tag{16}$$
It remains to find eigenvalues $J(\alpha^2)$ of equation (16) and boundary conditions (13).

It is convenient to use U as the independent variable. It can then be shown that
$$(1-U^2)\phi_{UU} - 2U\phi_U + \left\{2 - \frac{\alpha^2}{1-U^2} + \frac{J}{U^2(1-U^2)}\right\}\phi = 0, \tag{17}$$
and
$$\alpha\phi(U) = 0 \quad \text{at } U = \pm 1, \tag{18}$$
where the subscript U denotes differentiation with respect to U. Equation (17) has four regular singularities, at -1, 0, 1 and ∞. In fact, ϕ can be represented by the Riemann symbol
$$P\left\{\begin{matrix} -1 & 0 & -1 & \infty & \\ \tfrac{1}{2}\mu & \tfrac{1}{2}(1+\lambda) & \tfrac{1}{2}\mu & 2 & U \\ -\tfrac{1}{2}\mu & \tfrac{1}{2}(1-\lambda) & -\tfrac{1}{2}\mu & -1 & \end{matrix}\right\}, \tag{19}$$

where

$$\lambda = (1-4J)^{1/2}, \qquad \mu = +(\alpha^2-J)^{1/2}, \tag{20}$$

unless $J = 0$, in which case the symbol is

$$P\begin{bmatrix} -1 & 1 & \infty & \\ \frac{1}{2}\alpha & \frac{1}{2}\alpha & 2 & U \\ -\frac{1}{2}\alpha & -\frac{1}{2}\alpha & -1 & \end{bmatrix}.$$

Consider the special case $J = 0$ first. In this case a solution of Rayleigh's equation for a homogeneous fluid is sought. Equation (17) becomes the associated Legendre equation of degree one and order α. These Legendre functions are

$$\phi = (U-\alpha)\{(1+U)/(1-U)\}^{\frac{1}{2}\alpha} \quad \text{or} \quad (U+\alpha)\{(1-U)/(1+U)\}^{\frac{1}{2}\alpha}$$

in general. When $\alpha = 0$ or 1, these two solutions coincide; the second solutions are then

$$\phi = \tfrac{1}{2}U\log\frac{1+U}{1-U} - 1 \qquad (\alpha = 0),$$

$$\phi = (1-U^2)^{1/2}\left\{\tfrac{1}{2}\log\frac{1+U}{1-U} + \frac{U}{1-U^2}\right\} \qquad (\alpha = 1).$$

It can be seen that there are only two solutions satisfying the boundary conditions (18), namely

$$\left.\begin{aligned} &\alpha = 0, \qquad \phi = U = \tanh y, \\ \text{and} \qquad &\alpha = 1, \qquad \phi = (1-U^2)^{1/2} = \operatorname{sech} y. \end{aligned}\right\} \tag{21}$$

A natural way to solve the problem for small J would be to perturb these eigensolutions by use of a power series in J or some other convenient parameter. Before this can be done the singularities of ϕ must be removed. They may be divided out by defining a new dependent variable

$$\chi = (U+1)^{-\frac{1}{2}\mu}U^{-\frac{1}{2}(1+\lambda)}(U-1)^{-\frac{1}{2}\mu}\phi. \tag{22}$$

Then

$$\chi = P\begin{bmatrix} -1 & 0 & 1 & \infty & \\ 0 & 0 & 0 & \frac{5}{2}+\mu-\frac{1}{2}\lambda & U \\ -\mu & -\lambda & -\mu & -\frac{1}{2}+\mu-\frac{1}{2}\lambda & \end{bmatrix}$$

satisfies

$$\chi_{UU} + \left(\frac{1+\mu}{U-1} + \frac{1+\mu}{U+1} + \frac{1+\lambda}{U}\right)\chi_U + \frac{(\frac{5}{2}+\mu-\frac{1}{2}\lambda)(-\frac{1}{2}+\mu-\frac{1}{2}\lambda)}{(U+1)(U-1)}\chi = 0. \tag{23}$$

The boundary conditions are that χ is regular at $U = 0, \pm 1$.

Note that the coefficient of the last term of this equation is of the form $A(U+1)^{-1}(U-1)^{-1}$. The general form of this term for a function with four regular singularities at a, b, c, ∞ (one exponent at each of the finite singularities being zero) is $(AU+B)(U-a)^{-1}(U-b)^{-1}(U-c)^{-1}$, where A is the product of the exponents at infinity and B is some constant. Now the equation for ϕ, and therefore for χ, is even in U. This is why $B = 0$ in equation (23) and the last term takes its simple form.

It now appears that it is unnecessary to expand χ as a power series in J because an exact solution is given by

$$\chi = \text{constant}, \qquad (\tfrac{5}{2}+\mu-\tfrac{1}{2}\lambda)(-\tfrac{1}{2}+\mu-\tfrac{1}{2}\lambda) = 0.$$

Now $\alpha > 0$, and J is real. Therefore the real part of $\lambda = (1-4J)^{1/2}$ is less than unity in magnitude. But the sign of $\mu = +(\alpha^2-J)^{1/2}$ has been chosen so that μ has a non-negative real part. Therefore $\frac{5}{2}+\mu-\frac{1}{2}\lambda \neq 0$ in any real flow. Therefore

$$-\tfrac{1}{2}+\mu-\tfrac{1}{2}\lambda = 0. \tag{24}$$

On substituting for λ and μ from equations (20), it can be shown that this becomes

$$J = \alpha^2(1-\alpha^2). \tag{25}$$

This equation is plotted in figure 2. The eigenfunction is found, by putting χ = const. and $J = \alpha^2(1-\alpha^2)$ in equation (22), to be

$$\phi = (\operatorname{sech} y)^{\alpha^2}(\tanh y)^{1-\alpha^2}. \tag{26}$$

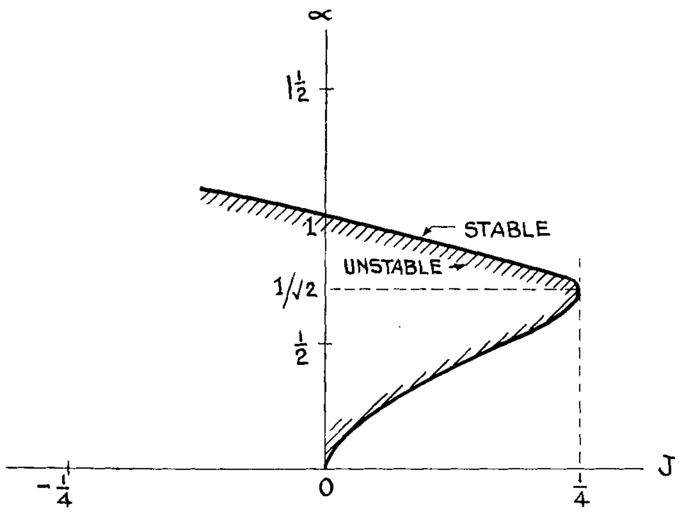

Figure 2. The curve of neutral stability $J = \alpha^2(1-\alpha^2)$.

This eigensolution, (25) and (26), joins up the only two solutions for $J = 0$. For each value of α, equation (25) gives a value of J for a neutral disturbance. The two roots of α^2 coincide if $J = \frac{1}{4}$ and are complex if $J > \frac{1}{4}$.

It may be noted that the horizontal velocity of the disturbance (i.e. $\phi' \exp(i\alpha x)$) is infinite in the critical plane where $y = 0$. This would be modified by inclusion of viscous terms as is met in the asymptotic theory of the Orr–Sommerfeld equation for large Reynolds number. However, the velocity flux is finite because $\phi' = O(1/y)$ as $y \to 0$. Also, the vertical velocity of the disturbance, $i\alpha\phi \exp(i\alpha x)$, is finite everywhere if J is non-negative.

5. Discussion of the stability criterion

The relation (25) for neutral stability in the (J, α)-plane is shown in figure 2. This is one part of the neutral curve, but other possible parts and the division of the plane into stable and unstable regions must be considered in order to see whether a wave is stable or unstable (i.e. whether $c_i < 0$ or $c_i > 0$) for any given values of J and α. The division of the plane will be made possible by the knowledge of the general stabilizing effect of the buoyancy forces and of the stability characteristics of a homogeneous fluid. In § 4 it was shown that if $J = 0$ and $c_i = 0$, then $c_r = 0$, and that the only two wave-numbers for neutral stability are $\alpha = 0, 1$. From the asymptotic viscous theory of the Orr–Sommerfeld equation it follows that the flow is stable for values (J, α) on the α-axis if $\alpha > 1$ and unstable if $\alpha < 1$. It may be concluded that the division of the plane into two simply-connected regions, the stable and the unstable, is as shown in figure 2. The possibility of a closed stable region to the left or unstable region to the right of the neutral curve must be admitted, but opposes the way in which gravity damps the energy of a disturbance and stabilizes a heterogeneous fluid.

On these grounds the critical Richardson number, the maximum value of J for which there can be instability, is $\frac{1}{4}$. This is the same as the critical value found by Goldstein (1931). This striking agreement, despite the different density and velocity profiles used, naturally raises the suggestion that it is not accidental. It might be thought, for instance, on noticing that $U = y$ near the critical plane $y = 0$ in both representations of the velocity profile in a shear layer, that the velocity near the critical plane effectively determines the stability of the flow. The value $J = \frac{1}{4}$ obtained by Goldstein (1931) comes from the order of a Bessel equation, the order being principally determined by the behaviour of U near the critical plane. However, this behaviour does not appear to be so important for the smooth profile $U = \tanh y$ because the effect of the other singularities of the stream function of the disturbance can be seen in the analysis of § 4. It has also been observed in § 3 that plane Couette flow must have a non-positive critical Richardson number, although the velocity has the same behaviour, $U \sim y$, near the critical plane. This shows that the value $J = \frac{1}{4}$ is determined by the unbounded shear flow as a whole rather than by its critical plane alone. Therefore, agreement, approximate or exact, is a justification of Goldstein's use of his three-layer flow to determine the stability criterion. The exactness of the agreement on a simple fraction $\frac{1}{4}$ appears to be accidental, and is presumably a consequence of the simplicity of the two flows being compared.

The condition $J > \frac{1}{4}$ for inviscid stability is also sufficient for viscous stability, provided that viscosity is solely a stabilizing effect in this flow of an unbounded heterogeneous fluid. The effects of density variation and viscosity are primarily separate, so it is possible to use flows of a homogeneous fluid as a guide. Heisenberg's criterion is known not to be valid for unbounded flows, but there has been little detailed calculation of their stability. Lessen (1949) calculated only part of the curve of neutral stability of a free boundary layer and found that a decrease of Reynolds number R caused an increase of stability in all his results.

Lessen (1949) did not explicitly calculate the velocity profile of the boundary layer from Blasius's equation, but Lock (1951) has done so. This permits a comparison of Lessen's results with those of § 4 after the dimensionless units have been related. The inviscid asymptote corresponds very closely to $\alpha = 1$, in agreement with the larger root of $J = \alpha^2(1-\alpha^2)$ for $J = 0$, which shows that the stability characteristics of an unbounded shear layer depend only weakly on the exact shape of the velocity profile. This gives the anticipated position of the ends of the curve of neutral stability in the (α, R)-plane as J increases from 0 to $\frac{1}{4}$. The upper branch is surmized to drop from $\alpha = 1$ to $\alpha = 1/\surd 2$; a lower branch (not calculated by Lessen) similarly rises from $\alpha = 0$ to $\alpha = 1/\surd 2$. When $J = \frac{1}{4}$, the branches coincide at $\alpha = 1/\surd 2$, and the flow is stable for all values of α and R. This indicates that $J > \frac{1}{4}$ is a sufficient condition of stability for a viscous shear layer between parallel streams of different densities.

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CORRIGENDUM

" Heat transfer from surfaces of non-uniform temperature ", by D. B. Spalding (*J. Fluid Mech.* **4**, 1958, 22).

Pages 29 and 30. In equations (13) and (14), and on figure 4, the coefficient 0·1 should be replaced by 0·2.